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SHORTER COMMUNICATION

LAMINAR FREE CONVECTION FROM A NON-ISOTHERMAL CONE AT LOW PRANDTL NUMBERS

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INTRODUCTION

IN MANY practical heat-transfer applications, the surface from which heat is being transferred is non-isothermal. However, most heat-transfer correlations available, due either to analysis or experiment, strictly apply for the uniform wall-temperature condition. Theoretical studies of heat transfer from non-isothermal surfaces in laminar free convection [1–3] for fluids with Prandtl numbers near unity, i.e. gases, have shown that the application of heat-transfer coefficients determined for uniform wall temperature conditions to variable wall-temperature situations can lead to significant errors in heat-transfer calculations. The purpose of this investigation is to study the non-isothermal wall problem in laminar free convection for low Prandtl number fluids. Axisymmetric flow about a vertical right circular cone with a power-law wall-temperature distribution was chosen for study. As shown in [3], the boundary-layer equations admit to a similarity transformation for this situation. Numerical solutions of the transformed boundary-layer equations have been obtained for Prandtl numbers typical of gases and liquid metals as well as for an inviscid fluid. Heat-transfer results for the isothermal cone have been reported [3–5] for gases.

FORMULATION AND SOLUTION

The boundary-layer equations are given and discussed in [3] where it was shown that a similarity transformation exists for wall temperature distributions

$$(T_w - T_\infty) = (T_0 - T_\infty)x^n.$$

The same notation is used as in reference 3. It is advantageous to utilize slightly altered variables from those in [3] for low Prandtl number fluid heat-transfer studies. Thus, introducing ζ and $f(\zeta)$ related to the corresponding variables η and $F(\eta)$ in [3] by

$$\zeta = \sqrt{(Pr)}\eta, \quad f(\zeta) = \sqrt{(Pr)}\,F(\eta) \tag{1}$$

the transformed boundary-layer equations and boundary conditions {equations (33), (34), and (15) of [3]} become

$$(Pr)f''' + \left(\frac{7+n}{4}\right)ff'' - \left(\frac{1+n}{2}\right)f'^2 + \theta = 0 \tag{2}$$

$$\theta'' + \left(\frac{7+n}{4}\right)f\theta' - nf'\theta = 0 \tag{3}$$

with

$$\left.\begin{array}{lll} f' = f = 0, & \theta = 1{\cdot}0, & \zeta = 0 \\ f' = 0, & \theta = 0, & \zeta = \infty. \end{array}\right\} \tag{4}$$

The primes in equations (2–4) denote differentiation with respect to ζ. In this form the contribution of the term representing viscous forces (f''') in equation (2) becomes small as Prandtl number decreases. Thus, as the Prandtl number approaches zero, the fluid behavior approaches that of an inviscid fluid with the buoyancy force just balanced by the inertia force. For $Pr = 0$, that is, free convection in an inviscid fluid, the requirement of no slip at the wall [$f'(0) = 0$] is no longer imposed. Le Fevre [6] obtained a result analogous to equation (2) with $Pr = 0$ in an investigation of the limiting value of heat transfer for a vertical isothermal plate.

Numerical solutions of equations (2) and (3) with the boundary conditions (4) have been obtained for Prandtl numbers of 0·001, 0·003, 0·01, 0·03, 0·1, 0·7, and 1·0 for the following values of n: 8, 4, 2, 1, 0·2, 0, −0·5. Solutions for an inviscid fluid ($Pr = 0$) for identical n values have also been determined. Table 1 lists the initial values $f''(0)$ and $\theta'(0)$ for the solutions obtained. These initial values are related to the friction drag and heat transfer.

RESULTS

Local Nusselt number

Introducing the altered variable of equation (1) into equation (43) of reference 3 results in the following expression for the local Nusselt number

$$\frac{Nu_x}{(Gr_x\,Pr^2)^{\frac{1}{4}}} = -\theta'(0) \tag{5}$$

Values of $\theta'(0)$ are given in Table 1 and were used to construct Figs. 1 and 2. In Fig. 1, this heat-transfer

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Table 1. Dimensionless velocity and temperature derivatives at the wall

n \ Pr	0		0·001		0·003		0·01	
	$-f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
−0·5	47·99	0·5920	56·03	0·5863	30·51	0·5780	15·36	0·5667
0	13·17	0·7916	47·96	0·7803	26·45	0·7669	13·55	0·7475
0·2	9·011	0·8567	45·52	0·8423	25·29	0·8280	13·01	0·8058
1·0	3·251	1·063	39·37	1·037	22·07	1·020	11·48	0·9886
2·0	1·728	1·245	34·98	1·210	19·70	1·189	10·31	1·149
4·0	1·032	1·491	30·23	1·444	17·08	1·416	8·987	1·364
8·0	0·7236	1·792	25·73	1·733	14·56	1·696	7·684	1·629

n \ Pr	0·03		0·1		0·7		1·0	
	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta''(0)$	$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$
−0·5	8·037	0·5493	3·830	0·5182	1·071	0·4304	0·8408	0·4094
0	7·185	0·7185	3·466	0·6683	0·9796	0·5392	0·7694	0·5104
0·2	6·928	0·7726	3·353	0·7159	0·9513	0·5730	0·7475	0·5148
1·0	6·174	0·9415	3·019	0·8663	0·8663	0·6777	0·6815	0·6389
2·0	5·582	1·089	2·749	0·9919	0·7962	0·7693	0·6270	0·7240
4·0	4·892	1·287	2·425	1·165	0·7096	0·8945	0·5596	0·8406
8·0	4·197	1·533	2·091	1·381	0·6172	1·053	0·4872	0·9889

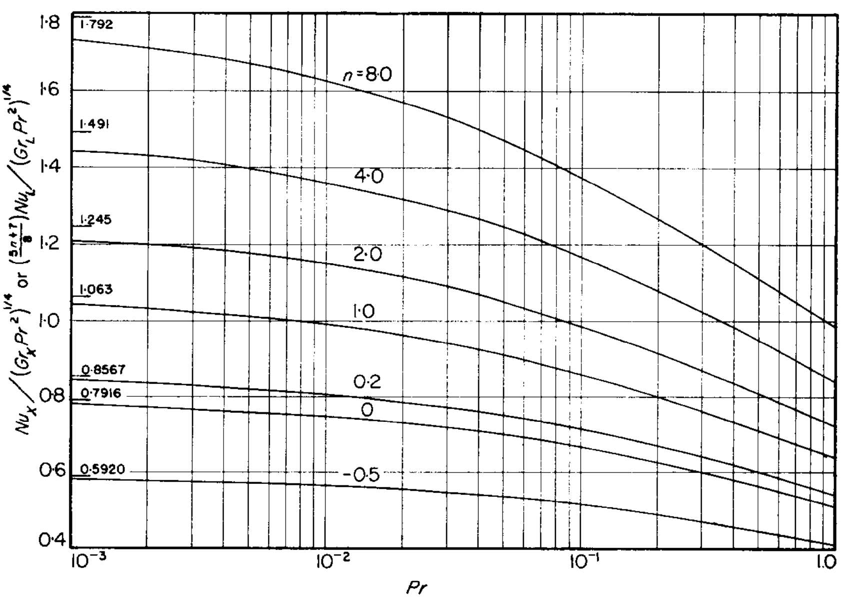

FIG. 1. Effect of Prandtl number on heat transfer for various n values.

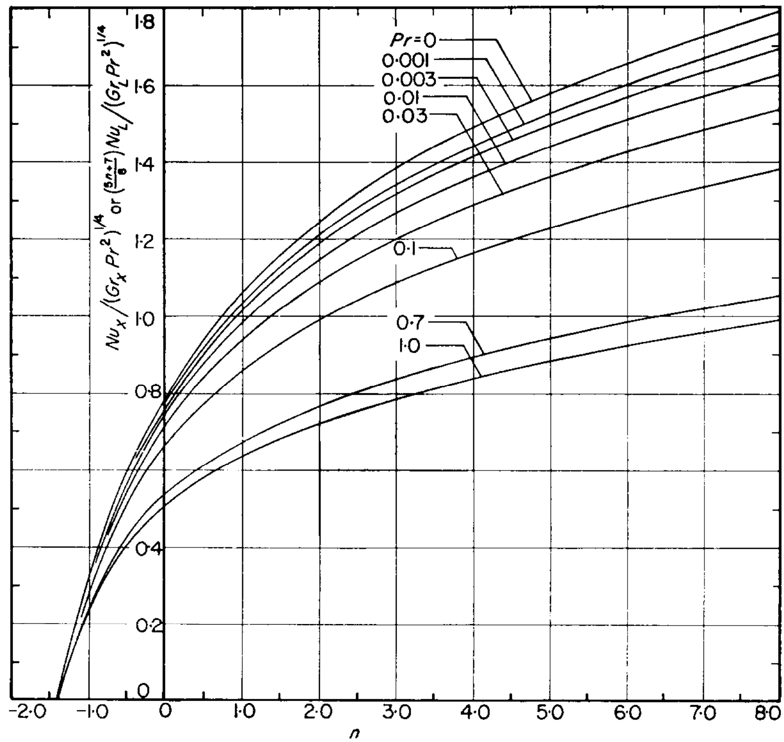

FIG. 2. Effect of n on heat transfer at low Prandtl number.

parameter is shown for the Prandtl number range 10^{-3} to 1·0 with n taking on values from $-0{\cdot}5$ to 8. The inviscid fluid results are given along the ordinate at the lowest value of Prandtl number in the figure. These solutions provide reasonably accurate local Nusselt number values for liquid metals. For example, at $Pr = 0{\cdot}03$, the inviscid fluid results are 10 and 17 per cent higher than the corresponding boundary-layer results for $n = 0$ and 8 respectively. These discrepancies reduce to 3 and 6 per cent at $Pr = 0{\cdot}003$. In Fig. 2 emphasis is given to the dependence of the dimensionless local heat-transfer relation of equation (5) on n with Prandtl number as a parameter. For positive and increasing values of n, the local Nusselt number expression shows substantial increases above that of the isothermal wall at a specified Prandtl number with the extent of this difference increasing as the Prandtl number decreases. Also apparent from this figure is the zero value for heat transfer when $n = -7/5$ independent of the value of Prandtl number. This result may be verified by inspection of equation (3).

It is of considerable practical interest to inquire as to how well the local heat flux for the non-isothermal wall condition can be predicted by the local application of isothermal wall results. In situations where heat-transfer coefficients for variable wall temperature are not available, calculations of local wall heat flux, in all likelihood, would proceed on the basis of heat-transfer coefficients determined for isothermal wall conditions and the local wall to ambient temperature difference. Denoting results obtained in this manner as q_{iso}, the ratio of the local heat-flux value for the variable wall temperature situation q_{var} to q_{iso} is given by

$$\frac{q_{\text{var}}}{q_{\text{iso}}} = \frac{[-\theta'(0)]_n}{[-\theta'(0)]_{n=0}} \tag{6}$$

where $[-\theta'(0)]_n$ represents the dimensionless temperature derivative at $\zeta = 0$ for a particular n and $[-\theta'(0)]_{n=0}$ denotes a similar derivative for the isothermal case. The results obtained in this manner are shown in Fig. 3 for representative values of Prandtl number from 0 to 1·0. Again, in agreement with other investigators, large errors in local heat flux predictions may be incurred by the local application of heat-transfer coefficients determined for isothermal wall conditions for fluids with Prandtl number near unity; i.e. for gases. The use of such procedures for fluids in the liquid metal range is even less satisfactory.

Mean Nusselt number

Introducing the variables of equation (1) into equation (48) of reference 3 gives

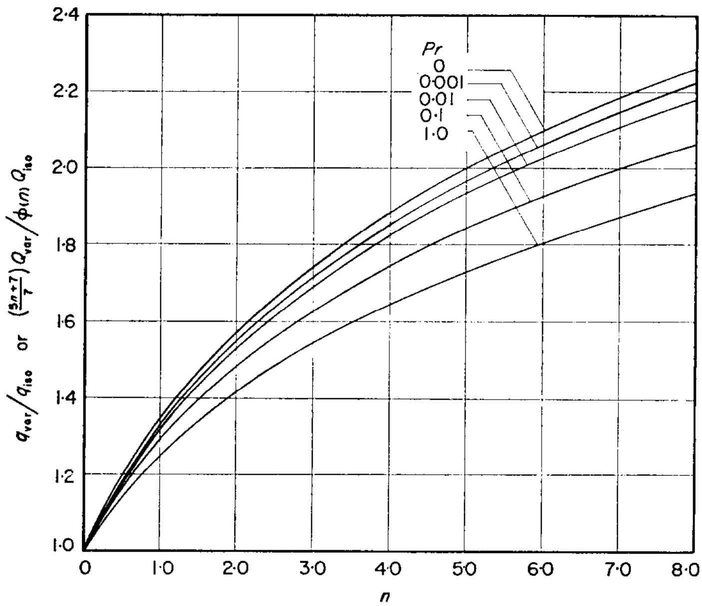

FIG. 3. Ratio of local heat fluxes and overall heat transfer.

$$\frac{Nu_L}{(Gr_L Pr^2)^{\frac{1}{4}}} = \frac{8}{5n + 7} [-\theta'(0)] \tag{7}$$

for $n > -7/5$. According to equations (6) and (7) the ordinates of Figs. 1 and 2 may also be interpreted as

$$\left(\frac{5n + 7}{8}\right) \frac{Nu_L}{(Gr_L Pr^2)^{\frac{1}{4}}}$$

Also of interest is the accuracy to which total heat transfer for variable wall temperature conditions may be predicted by the application of overall heat-transfer coefficients determined for the isothermal wall. Letting Q_{iso} denote the total heat transfer evaluated on the basis of an isothermal wall heat-transfer coefficient, we may write

$$Q_{iso} = \bar{h}_{iso} A(T_c - T_\infty) \tag{8}$$

where A is the cone lateral surface area and $T_c - T_\infty$ is some characteristic temperature difference for the variable wall temperature situation which is also used for the evaluation of Gr_L. With the definition of $\bar{h}$ from equation (47) of reference 3, the ratio of the total heat transfer for variable wall temperature Q_{var} to that given by equation (8) is

$$\frac{Q_{var}}{Q_{iso}} = \left(\frac{7}{5n + 7}\right) \phi(n) \left(\frac{q_{var}}{q_{iso}}\right). \tag{9}$$

where

$$\phi(n) = \left(\frac{T_o - T_\infty}{T_c - T_\infty}\right)^{5/4} \tag{10}$$

Thus, the ordinate of Fig. 3 may also be interpreted as

$$\frac{Q_{var}}{Q_{iso}} \left(\frac{5n + 7}{7}\right) \frac{1}{\phi(n)}.$$

The choice of a characteristic temperature difference, $T_c - T_\infty$, for the variable temperature wall conditions is quite arbitrary. One choice which comes to mind because of the definition of $\bar{h}$ is to use the temperature difference at $X = L$, that is $T_c - T_\infty = T_o - T_\infty$. For this selection $\phi(n) = 1$ and the total heat transfer calculated on the basis of equation (8) is overestimated for $n > 0$ and underestimated for $n < 0$. As an example of magnitude of this error, the heat transfer is only sixty-three per cent of that predicted by Q_{iso} for $n = 2$ and Prandtl number 0·01. Even larger inaccuracies occur for larger n values and higher Prandtl numbers.

A second choice for the characteristic temperature difference suggested by simplicity is the mean surface temperature difference. For this selection $\phi(n) = (n + 1)^{5/4}$ and thus heat transfer is underestimated for $n > 0$ and overestimated for $n < 0$. For the same values of n and Prandtl number discussed above, the total heat transfer is almost 2·5 times that calculated by equation (8). Again, the errors increase with increasing departure from isothermal wall temperature conditions. No simple method of choosing the characteristic temperature difference has been found which will lead to accurate predictions of total heat-transfer rates for variable wall temperature conditions by the use of average heat-transfer coefficients for the isothermal wall.

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